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# On simulating a medium with special reflecting properties by Lobachevsky geometry (One exactly solvable electromagnetic problem)

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Lobachewsky geometry simulates a medium with special constitutive relations,  $D^i = \epsilon_0 \epsilon^{ik} E^k$ ,  $B^i = \mu_0 \mu^{ik} H^k$ , where two matrices coincide:  $\epsilon^{ik}(x) = \mu^{ik}(x)$ . The situation is specified in quasi-cartesian coordinates  $(x, y, z)$ . Exact solutions of the Maxwell equations in complex 3-vector  $\mathbf{E} + i\mathbf{B}$  form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field  $U(z) = U_0 e^{2z}$ . In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient  $R = 1$ ; in electrodynamic context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency  $\omega$ ,  $k_1^2 + k_2^2$ , and the curvature radius  $\rho$ .

## 1 Introduction

An aim of the present paper is to obtain exact solutions of the Maxwell equations in 3-dimensional Lobachevsky space  $H_3$ . A coordinate system used is one from the list given by Olevsky [1], which generalizes Cartesian coordinate in flat Euclidean space.

To treat Maxwell equations we make use of complex representation of these according to the known approach by Riemann–Silberstein–Oppenheimer–Majorana [2, 3, 4, 5] (see also in [6 – 30]), extended to curved space-time models in the frames of tetrad formalism of Tetrode–Weyl–Fock–Ivanenko [31, 32, 33]; see also in [34]). On the base of this technique, new exact solutions of the type of extended plane wave in Lobachevsky space have been constructed explicitly. These may be interesting in the cosmological sense; besides, they may be interesting in the context of geometric simulating electromagnetic field in a special medium [35], [34].

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## 2 Cartezian coordinates in Lobachevsky space

In Olevsky paper [1], under the number 2 the following coordinate system in Lobachevsky space  $H_3$  is specified

$$x^a = (t, x, y, z) , \quad dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 , \quad (1)$$

the element of volume is given by

$$dV = \sqrt{-g} \, dx dy dz = e^{-2z} dx dy dz , \quad x, y, z \in (-\infty, +\infty) ;$$

the magnitude and sign of the  $z$  are substantial, in particular, when dealing with localization, for example, the energy of the field

$$dW = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)dV = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) e^{-2z} dx dy dz . \quad (2)$$

It is helpful to have at hand some detail of the parametrization of the model  $H_3$  by  $x, y, z$ . It is known that this model can be identified with a branch of hyperboloid in 4-dimension flat space

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2 , \quad u_0 = +\sqrt{\rho^2 + \mathbf{u}^2} .$$

Coordinate in use,  $x, y, z$ , are referred to  $u_a$  by relations

$$\begin{aligned} u_1 &= x e^{-z} , \quad u_2 = y e^{-z} , \\ u_3 &= \frac{1}{2}[(e^z - e^{-z}) + (x^2 + y^2)e^{-z}] , \\ u_0 &= \frac{1}{2}[(e^z + e^{-z}) + (x^2 + y^2)e^{-z}] . \end{aligned} \quad (3)$$

It is convenient to employ 3-dimensional Poincaré realization for Lobachevsky space as inside part of 3-sphere

$$q_i = \frac{u_i}{u_0} = \frac{u_i}{\sqrt{\rho^2 + u_1^2 + u_2^2 + u_3^2}} , \quad q_i q_i < +1 . \quad (4)$$

Quasi-Cartesian coordinates  $(x, y, z)$  are referred to  $q_i$  as follows

$$\begin{aligned} q_1 &= \frac{2x}{x^2 + y^2 + e^{2z} + 1} , \\ q_2 &= \frac{2y}{x^2 + y^2 + e^{2z} + 1} , \\ q_3 &= \frac{x^2 + y^2 + e^{2z} - 1}{z^2 + y^2 + e^{2z} + 1} ; \end{aligned} \quad (5)$$

Inverses to (5) relations are

$$x = \frac{q_1}{1 - q_3} , \quad y = \frac{q_2}{1 - q_3} , \quad e^z = \frac{\sqrt{1 - q^2}}{1 - q_3} . \quad (6)$$

In particular, note that on the axis  $q_1 = 0, q_2 = 0, q \in (-1, +1)$  relations (6) assume the form

$$x = 0, \quad y = 0, \quad e^z = \sqrt{\frac{1+q_3}{1-q_3}}.$$

that is

$$\begin{aligned} q_3 &\longrightarrow +1, & e^z &\longrightarrow +\infty, & z &\longrightarrow +\infty; \\ q_3 &\longrightarrow -1, & e^z &\longrightarrow +0, & z &\longrightarrow -\infty. \end{aligned} \quad (7)$$

Solutions of the Maxwell equation, constructed bellow, can be of interest in the context of description of electromagnetic waves in special media, because the Lobachevsky geometry simulates effectively a definite special medium [36], inhomogeneous along the axis  $z$ . Effective electric permittivity tensor  $\epsilon^{ik}(x)$  is given by

$$\epsilon^{ik}(x) = -\sqrt{-g} g^{00}(x) g^{ik}(x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2z} \end{vmatrix}, \quad (8)$$

whereas the corresponding effective magnetic permittivity tensor is

$$(\mu^{-1})^{ik}(x) = \sqrt{-g} \begin{vmatrix} g^{22}g^{33} & 0 & 0 \\ 0 & g^{33}g^{11} & 0 \\ 0 & 0 & g^{11}g^{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2z} \end{vmatrix}. \quad (9)$$

In explicit form, effective constitutive relations (the system  $SI$  is used) are

$$D^i = \epsilon_0 \epsilon^{ik} E_k, \quad B_i = \mu_0 \mu^{ik} H^k, \quad (10)$$

note that two matrices coincide:  $\epsilon^{ik}(x) = \mu^{ik}(x)$ .

### 3 Tetrads and Maxwell equations in complex form

In the coordinate (1), let us introduce a tetrad

$$e_{(a)}^\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & e^z & 0 & 0 \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad e_{(a)\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-z} & 0 & 0 \\ 0 & 0 & -e^{-z} & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \quad (11)$$

One should find Christoffel symbols; some of them evidently vanish:  $\Gamma_{\beta\sigma}^0 = 0$ ,  $\Gamma_{00}^i = 0$ ,  $\Gamma_{0j}^i = 0$ , remaining ones are determined by relations

$$\Gamma_{jk}^x = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}, \quad \Gamma_{jk}^y = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix}, \quad \Gamma_{jk}^z = \begin{vmatrix} e^{-2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Ricci rotation coefficients are (only not vanishing ones are written down)

$$\gamma_{311} = -1, \quad \gamma_{232} = 1.$$

Using the notation [34]

$$\begin{aligned} e_{(0)}^\rho \partial_\rho &= \partial_{(0)} = \partial_t, & e_{(1)}^\rho \partial_\rho &= \partial_{(1)} = e^z \partial_x, \\ e_{(2)}^\rho \partial_\rho &= \partial_{(2)} = e^z \partial_y, & e_{(3)}^\rho \partial_\rho &= \partial_{(3)} = \partial_z, \\ \mathbf{v}_0 &= (\gamma_{010}, \gamma_{020}, \gamma_{030}) \equiv 0, & \mathbf{v}_1 &= (\gamma_{011}, \gamma_{021}, \gamma_{031}) \equiv 0, \\ \mathbf{v}_2 &= (\gamma_{0120}, \gamma_{022}, \gamma_{032}) \equiv 0, & \mathbf{v}_3 &= (\gamma_{013}, \gamma_{023}, \gamma_{033}) \equiv 0, \\ \mathbf{p}_0 &= (\gamma_{230}, \gamma_{310}, \gamma_{120}) = 0, & \mathbf{p}_1 &= (\gamma_{231}, \gamma_{311}, \gamma_{121}) = (0, -1, 0), \\ \mathbf{p}_2 &= (\gamma_{232}, \gamma_{312}, \gamma_{122}) = (1, 0, 0), & \mathbf{p}_3 &= (\gamma_{233}, \gamma_{313}, \gamma_{123}) = 0; \end{aligned}$$

the Maxwell equations in the complex matrix form [34] read

$$\left[ \alpha^k \partial_{(k)} + \mathbf{sv}_0 + \alpha^k \mathbf{sp}_k - i (\partial_{(0)} + \mathbf{sp}_0 - \alpha^k \mathbf{sv}_k) \right] \begin{vmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{vmatrix} = 0; \quad (12)$$

in the used retrad it assumes the form

$$(-i\partial_t + \alpha^1 e^z \partial_x + \alpha^2 e^z \partial_y + \alpha^3 \partial_z - \alpha^1 s_2 + \alpha^2 s_1) \begin{vmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{vmatrix} = 0. \quad (13)$$

Matrices involved in (13) are

$$\begin{aligned} \alpha^1 &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, & \alpha^2 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \\ \alpha^3 &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, & s^1 &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, & s^2 &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}. \end{aligned}$$

## 4 Separation of the variables

Let us use the substitution

$$\begin{vmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{vmatrix} = e^{-i\omega t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} 0 \\ \mathbf{f}(z) \end{vmatrix}. \quad (14)$$

correspondingly eq. (14) gives

$$\left( -\omega + \alpha^1 e^z i k_1 + \alpha^2 e^z i k_2 + \alpha^3 \frac{d}{dz} - \alpha^1 s_2 + \alpha^2 s_1 \right) \begin{vmatrix} 0 \\ f_1(z) \\ f_2(z) \\ f_3(z) \end{vmatrix} = 0. \quad (15)$$

After simple calculation, we derive a first order system for  $f_i$ :

$$\begin{aligned}
ik_1 e^z f_1 + ik_2 e^z f_2 + \left(\frac{d}{dz} - 2\right)f_3 &= 0, \\
-\omega f_1 - \left(\frac{d}{dz} - 1\right)f_2 + ik_2 e^z f_3 &= 0, \\
-\omega f_2 + \left(\frac{d}{dz} - 1\right)f_1 - ik_1 e^z f_3 &= 0, \\
-\omega f_3 - e^z ik_2 f_1 + ik_1 e^z f_2 &= 0.
\end{aligned} \tag{16}$$

Allowing three last equations in the first one, we get an identity  $0 = 0$ . So, there exist only three independent equations (below the notation  $k_1 = a, k_2 = b$  is used):

$$\begin{aligned}
\omega f_3 &= -ib e^z f_1 + ia e^z f_2, \\
\omega f_1 &= -\left(\frac{d}{dz} - 1\right)f_2 + ib e^z f_3, \\
\omega f_2 &= +\left(\frac{d}{dz} - 1\right)f_1 - ia e^z f_3,
\end{aligned} \tag{17}$$

With substitutions  $f_1 = e^z F_1(z)$ ,  $f_2 = e^z F_2(z)$ , eqs. (17) give

$$\begin{aligned}
\omega f_3 &= -ib e^{2z} F_1 + ia e^{2z} F_2, \\
\omega F_1 &= -\frac{d}{dz} F_2 + ib f_3, \\
\omega F_2 &= +\frac{d}{dz} F_1 - ia f_3.
\end{aligned} \tag{18}$$

There exist a particular case readily treatable, when  $a = 0$ ,  $b = 0$ ,  $f_3 = 0$ :

$$\begin{aligned}
\omega F_1 &= -\frac{d}{dz} F_2, \quad \omega F_2 = +\frac{d}{dz} F_1 \implies \\
F_1(z) &= e^{\pm i\omega z}, \quad F_2 = \pm i e^{\pm i\omega z},
\end{aligned} \tag{19}$$

which gives

$$\Phi^\pm = \begin{vmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{vmatrix} = e^{-i\omega t} e^z \begin{vmatrix} 0 \\ e^{\pm i\omega z} \\ \pm i e^{\pm i\omega z} \\ 0 \end{vmatrix} \tag{20}$$

or (let it be  $\varphi^{(\pm)} = \omega t \mp \omega z$ )

$$\begin{aligned}
E_1^{(\pm)} + iB_1^{(\pm)} &= \cos(\omega t \mp \omega z) - i \sin(\omega t \mp \omega z), \\
E_2^{(\pm)} + iB_2^{(\pm)} &= \pm \sin(\omega t \mp \omega z) \pm i \cos(\omega t \mp \omega z).
\end{aligned} \tag{21}$$

It is easily checked the known presupposed property  $\mathbf{E}^{(\pm)} \times \mathbf{B}^{(\pm)} = \pm \mathbf{e}_z$ .

Let us turn back to the generale case (18), from the first equation it follows

$$f_3 = \frac{-ib}{\omega} e^{2z} F_1 + \frac{ia}{\omega} e^{2z} F_2, \tag{22}$$

and further we get a system for  $F_1$  and  $F_2$

$$\begin{aligned} \left(\frac{d}{dz} + \frac{ab e^{2z}}{\omega}\right) F_2 &= \frac{b^2 e^{2z} - \omega^2}{\omega} F_1, \\ \left(\frac{d}{dz} - \frac{ab e^{2z}}{\omega}\right) F_1 &= \frac{\omega^2 - a^2 e^{2z}}{\omega} F_2. \end{aligned} \quad (23)$$

With the help of a new variable  $e^z = \sqrt{\omega} Z$ , two last are written as

$$\begin{aligned} Z \left(\frac{d}{dZ} + ab Z\right) F_2 &= +(b^2 Z^2 - \omega) F_1, \\ Z \left(\frac{d}{dZ} - ab Z\right) F_1 &= -(a^2 Z^2 - \omega) F_2. \end{aligned} \quad (24)$$

This system can be solved straightforwardly in terms of Heun confluent functions. Indeed, from (24) it follows a second order differential equation for  $F_1$

$$\frac{d^2 F_1}{dZ^2} - \frac{a^2 Z^2 + \omega}{Z(a^2 Z^2 - \omega)} \frac{dF_1}{dZ} + \left[ \frac{\omega^2}{Z^2} + \frac{2ab\omega}{a^2 Z^2 - \omega} - (a^2 + b^2)\omega \right] F_1 = 0, \quad (25)$$

here we note additional singular point at  $Z = \pm\sqrt{\omega}/a$ . With the new variable, we get

$$\begin{aligned} y = \frac{a^2 Z^2}{\omega}, \quad \frac{d^2 F_1}{dy^2} + \left[ \frac{1}{y} - \frac{1}{y-1} \right] \frac{dF_1}{dy} \\ + \left[ \frac{\omega^2}{4y^2} - \frac{2ab\omega + (a^2 + b^2)\omega^2}{4a^2 y} + \frac{b\omega}{2a(y-1)} \right] F_1 = 0. \end{aligned} \quad (26)$$

from whence or with the substitution  $F_1(y) = y^c g_1(y)$  we arrive at

$$\begin{aligned} \frac{d^2 g_1}{dy^2} + \left[ \frac{2c+1}{y} - \frac{1}{y-1} \right] \frac{dg_1}{dy} + \left[ \frac{\omega^2/4 + c^2}{y^2} \right. \\ \left. + \frac{2c - \omega^2/2 - b\omega/a - b^2\omega^2/(2a^2)}{2y} + \frac{-2c + b\omega/a}{2(y-1)} \right] g_1 = 0. \end{aligned} \quad (27)$$

When  $c = \pm i\omega/2$ , eq. (27) is simplified

$$\begin{aligned} \frac{d^2 g_1}{dy^2} + \left[ \frac{2c+1}{y} - \frac{1}{y-1} \right] \frac{dg_1}{dy} \\ + \left[ \frac{2c - \omega^2/2 - b\omega/a - b^2\omega^2/(2a^2)}{2y} + \frac{-2c + b\omega/a}{2(y-1)} \right] g_1 = 0 \end{aligned}$$

which can be identified with confluent Heun function

$$\begin{aligned} H(\alpha, \beta, \gamma, \delta, \eta, z), \quad \frac{d^2 H}{dz^2} + \left[ \alpha + \frac{1+\beta}{z} + \frac{1+\gamma}{z-1} \right] \frac{dH}{dz} \\ + \left[ \frac{1}{2} \frac{\alpha + \alpha\beta - \beta\gamma - \beta - \gamma - 2\eta}{z} + \frac{1}{2} \frac{\alpha\gamma + \beta + \alpha + 2\eta + 2\delta + \beta\gamma + \gamma}{z-1} \right] H = 0 \end{aligned} \quad (28)$$

with parameters

$$\begin{aligned} \alpha &= 0, & \beta &= 2c, & \gamma &= -2, & \delta &= -\frac{1}{4} \frac{(a^2 + b^2) \omega^2}{a^2}, \\ \eta &= \frac{1}{4} \frac{2ab\omega + (a^2 + b^2)\omega^2 + 4a^2}{a^2}, & F_1 &= y^{\pm i\omega/2} H(\alpha, \beta, \gamma, \delta, \eta, y). \end{aligned} \quad (29)$$

Below we will develop a method that makes possible to construct solutions of the system (23) in more simple functions, solution of the Bessel equation.

## 5 Additional studying of the system

Let us perform a special transformation in (23) (suppose  $(\alpha n - \beta m) = 1$ )

$$\begin{aligned} F_1 &= \alpha G_1 + \beta G_2, & F_2 &= m G_1 + n G_2; \\ G_1 &= n F_1 - \beta F_2, & G_2 &= -m F_1 + \alpha F_2. \end{aligned} \quad (30)$$

Combining equations from (23), we get

$$\begin{aligned} n Z \left( \frac{d}{dZ} - ab Z \right) F_1 - \beta Z \left( \frac{d}{dZ} + ab Z \right) F_2 &= -n (a^2 Z^2 - \omega) F_2 - \beta (b^2 Z^2 - \omega) F_1, \\ -m Z \left( \frac{d}{dZ} - ab Z \right) F_1 + \alpha Z \left( \frac{d}{dZ} + ab Z \right) F_2 &= m (a^2 Z^2 - \omega) F_2 + \alpha (b^2 Z^2 - \omega) F_1, \end{aligned}$$

from whence it follows

$$\begin{aligned} Z \frac{d}{dZ} G_1 - Z^2 ab (nF_1 + \beta F_2) &= -Z^2 (na^2 F_2 + \beta b^2 F_1) + \omega (nF_2 + \beta F_1), \\ Z \frac{d}{dZ} G_2 + Z^2 ab (mF_1 + \alpha F_2) &= Z^2 (ma^2 F_2 + \alpha b^2 F_1) - \omega (mF_2 + \alpha F_1). \end{aligned} \quad (31)$$

Taking into account (30), eqs. (31) reduce to

$$\begin{aligned} \left[ Z \frac{d}{dZ} - Z^2 ab (n\alpha + \beta m) + Z^2 (a^2 mn + b^2 \alpha \beta) - \omega (nm + \alpha \beta) \right] G_1 \\ = [-Z^2 (an - b\beta)^2 + \omega (n^2 + \beta^2)] G_2, \\ \left[ Z \frac{d}{dZ} + Z^2 ab (m\beta + n\alpha) - Z^2 (a^2 mn + b^2 \alpha \beta) + \omega (nm + \alpha \beta) \right] G_2 \\ = [Z^2 (am - b\alpha)^2 - \omega (m^2 + \alpha^2)] G_1. \end{aligned} \quad (32)$$

Let us impose additional restriction (there exist two possibilities):

$$\begin{aligned} an - b\beta = 0 &\implies \frac{\beta}{n} = \frac{a}{b}, \\ \left[ Z \frac{d}{dZ} - Z^2 ab (n\alpha + \beta m) + Z^2 (a^2 mn + b^2 \alpha \beta) - \omega (nm + \alpha \beta) \right] G_1 \\ = +\omega (n^2 + \beta^2) G_2, \\ \left[ Z \frac{d}{dZ} + Z^2 ab (m\beta + n\alpha) - Z^2 (a^2 mn + b^2 \alpha \beta) + \omega (nm + \alpha \beta) \right] G_2 \\ = [Z^2 (am - b\alpha)^2 - \omega (m^2 + \alpha^2)] G_1. \end{aligned} \quad (33)$$

or

$$\begin{aligned}
am - b; \alpha = 0 & \implies \frac{\alpha}{m} = \frac{a}{b}, \\
\left[ Z \frac{d}{dZ} - Z^2 ab(n\alpha + \beta m) + Z^2(a^2 mn + b^2 \alpha \beta) - \omega(nm + \alpha \beta) \right] G_1 \\
&= [-Z^2(an - b\beta)^2 + \omega(n^2 + \beta^2)] G_2, \\
\left[ Z \frac{d}{dZ} + Z^2 ab(m\beta + n\alpha) - Z^2(a^2 mn + b^2 \alpha \beta) + \omega(nm + \alpha \beta) \right] G_2 \\
&= -\omega(m^2 + \alpha^2) G_1.
\end{aligned} \tag{34}$$

The two variant are equivalent each other, for definiteness we will use the variant (33). It can be presented in more symmetrical form

$$\begin{aligned}
F_1 &= \alpha G_1 + \beta G_2 = +\frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2, \\
F_2 &= m G_1 + n G_2 = -\frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2;
\end{aligned} \tag{35}$$

at this eqs. (18) assume the form

$$\begin{aligned}
\left[ Z \frac{d}{dZ} - Z^2 ab \frac{b^2 - a^2}{b^2 + a^2} + Z^2 ab \frac{b^2 - a^2}{b^2 + a^2} - \omega \left( -\frac{ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2} \right) \right] G_1 \\
&= +\omega \left( \frac{b^2}{a^2 + b^2} + \frac{a^2}{a^2 + b^2} \right) G_2, \\
\left[ Z \frac{d}{dZ} + Z^2 ab \frac{b^2 - a^2}{a^2 + b^2} - Z^2 ab \frac{b^2 - a^2}{a^2 + b^2} + \omega \left( -\frac{ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2} \right) \right] G_2 \\
&= \left[ Z^2 \left( -\frac{a^2}{\sqrt{a^2 + b^2}} - \frac{b^2}{\sqrt{a^2 + b^2}} \right)^2 - \omega \left( \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} \right) \right] G_1,
\end{aligned}$$

that is

$$Z \frac{d}{dZ} G_1 = \omega G_2, \quad Z \frac{d}{dZ} G_2 = [Z^2(a^2 + b^2) - \omega] G_1. \tag{36}$$

From (36) we derive a second order equation for  $G_1$

$$\left( Z^2 \frac{d^2}{dZ^2} + Z \frac{d}{dZ} + \omega^2 - \omega(a^2 + b^2)Z^2 \right) G_1 = 0. \tag{37}$$

To understand better the physical meaning of the equation (37), it is convenient to translate the equation to variable  $z$ , then it reads

$$e^z = \sqrt{\omega} Z, \quad \left( \frac{d^2}{dz^2} + \omega^2 - (a^2 + b^2)e^{2z} \right) G_1 = 0. \tag{38}$$



It can be associated with the Schrödinger equation

$$\left( \frac{d^2}{dz^2} + \epsilon - U(z) \right) \varphi(z) = 0 \quad (39)$$

with potential function  $U(z) = (a^2 + b^2)e^{2z}$ , and an effective force acting on the left  $F_z = -2(a^2 + b^2)e^{2z}$ . Note that when  $a = k_1 = 0$ ,  $b = k_2 = 0$ , the effective force vanishes. The corresponding quantum-mechanical system can be illustrated by Fig.1.

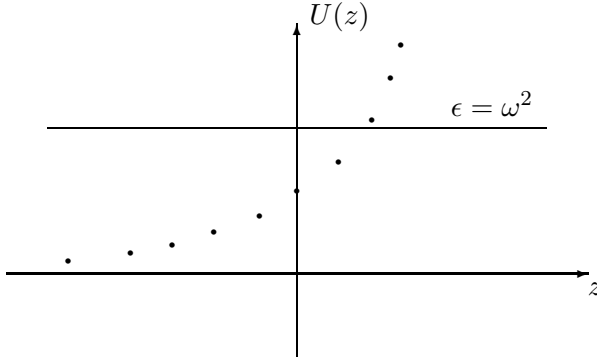


Figure 1: Effective potential curve

Therefore, we should expect properties of the electromagnetic solutions similar to those existing in the associated quantum-mechanical problem.

Let us turn back to eq. (37) – in the variable

$$x = i \sqrt{\omega(a^2 + b^2)} Z = i \sqrt{a^2 + b^2} e^z$$

it assumes the form of the Bessel equation

$$\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{\omega^2}{x^2} \right) G_1 = 0. \quad (40)$$

The first order system (36) in variable  $x$  takes the form

$$x \frac{d}{dx} G_1 = \omega G_2, \quad x \frac{d}{dx} G_2 = -\frac{\omega^2 + x^2}{\omega} G_1. \quad (41)$$

A second order equation for  $G_2$  reads

$$\left[ \frac{d^2}{dx^2} + \left( \frac{1}{x} - \frac{2x}{\omega^2 + x^2} \right) \frac{d}{dx} + \frac{x^2 + \omega^2}{x^2} \right] G_2 = 0. \quad (42)$$

Note that substituting (35)

$$F_1 = \frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2, \quad F_2 = -\frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2$$

into (22), we get

$$f_3 = \frac{e^{2z}}{\omega} (-ib F_1 + ia F_2) = \frac{\sqrt{a^2 + b^2}}{i \omega} G_1. \quad (43)$$

## 6 Asymptotic behavior of solutions

Mostly used for Bessel equation [37] are solutions

in Bessel's functions

$$G_1^I(x) = J_{+i\omega}(x) , \quad G_1^{II}(x) = J_{-i\omega}(x) ; \quad (44)$$

in Hankel's functions

$$\begin{aligned} G_1^I(x) &= H_{+i\omega}^{(1)}(x) , & G_1^{II}(x) &= H_{+i\omega}^{(2)}(x) , \\ I' \quad G_1(x) &= H_{-i\omega}^{(1)}(x) , & II' \quad G_1(x) &= H_{-i\omega}^{(2)}(x) ; \end{aligned} \quad (45)$$

note that  $H_{-i\omega}^{(1)}(x) = e^{-\omega\pi} H_{i\omega}^{(2)}(x)$ , so the primed cases  $I', II'$  coincide respectively with  $II, I$  and by this reason will not be considered below;

in Neyman functions

$$G_1^I(x) = N_{+i\omega}(x) , \quad G_1^{II}(x) = N_{-i\omega}(x) . \quad (46)$$

For shortness, below the notation  $+\sqrt{a^2+b^2} = 2\sigma$  is used. First, let us consider solutions in Bessel's functions [37] when

$$z \rightarrow -\infty, \quad x = i\sigma e^z \rightarrow i0 ,$$

$$\begin{aligned} G_1^I(x) &= J_{+i\omega}(x) = \frac{1}{\Gamma(1+i\omega)} \left(\frac{x}{2}\right)^{+i\omega} = \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} , \\ G_1^{II}(x) &= J_{-i\omega}(x) = \frac{1}{\Gamma(1-i\omega)} \left(\frac{x}{2}\right)^{-i\omega} = \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} . \end{aligned} \quad (47)$$

In the region  $z \rightarrow +\infty$ , ( $x = i\sigma e^z = iX \rightarrow i\infty$ ), using the known asymptotic formula [37]

$$J_{i\omega}(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - (i\omega + \frac{1}{2})\frac{\pi}{2}\right) ,$$

we get

$$\begin{aligned} G_1^I(z \rightarrow \infty) &= J_{+i\omega}(z \rightarrow \infty) \sim e^{i\pi/4} \sqrt{\frac{1}{2\pi iX}} e^{-\omega\pi/2} e^{+X} , \\ G_1^{II}(z \rightarrow \infty) &= J_{-i\omega}(z \rightarrow \infty) \sim e^{i\pi/4} \sqrt{\frac{1}{2\pi iX}} e^{+\omega\pi/2} e^{+X} . \end{aligned} \quad (48)$$

Let us consider solutions in Hankel's functions [37], determined in terms of  $J_{\pm i\omega}(x)$  as follows

$$\begin{aligned} H_{i\omega}^{(1)}(x) &= +\frac{i}{\sin(i\omega\pi)} (e^{\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)) , \\ H_{i\omega}^{(2)}(x) &= -\frac{i}{\sin(i\omega\pi)} (e^{-\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)) . \end{aligned} \quad (49)$$

so that  $z \rightarrow -\infty$ ,  $x \rightarrow i0$ ,

$$\begin{aligned} G_1^I(x) &= H_{i\omega}^{(1)}(x) = +\frac{i}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right), \\ G_1^{II}(x) &= H_{i\omega}^{(2)}(x) = -\frac{i}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{+i\omega z} \right). \end{aligned} \quad (50)$$

Behavior of them when  $z \rightarrow +\infty$  is governed the known relation [37]

$$\begin{aligned} H_{i\omega}^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[ +i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right], \\ H_{i\omega}^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[ -i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right]; \end{aligned}$$

from whence it follows

$$z \rightarrow +\infty, \quad x = iX \rightarrow i\infty,$$

$$\begin{aligned} G_1^I(x) &= H_{i\omega}^{(1)}(x) \sim e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \\ G_1^{II}(x) &= H_{i\omega}^{(2)}(x) \sim e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}. \end{aligned} \quad (51)$$

Let us consider interpretation of the first type solution: this wave goes from the left, then it is partly reflected and partly goes forward through an effective potential barrier but gradually damping as  $z$  rises. The corresponding reflection coefficient is determined as follows

$$G(z) \sim M_+^I e^{+i\omega z} + M_-^I e^{-i\omega z}, \quad R = \frac{|M_-^I|^2}{|M_+^I|^2}. \quad (52)$$

Taking into account identities

$$\begin{aligned} (i\sigma)^{+i\omega} &= (e^{i\pi/2} e^{\ln \sigma})^{+i\omega} = e^{-\omega\pi/2} e^{+i\omega \ln \sigma}, \\ (i\sigma)^{-i\omega} &= (e^{i\pi/2} e^{\ln \sigma})^{-i\omega} = e^{+\omega\pi/2} e^{-i\omega \ln \sigma}; \end{aligned} \quad (53)$$

we derive

$$\begin{aligned} |M_+^I|^2 &= \frac{1}{\sin(+i\omega\pi) \sin(-i\omega\pi)} \frac{e^{+\omega\pi}}{\Gamma(1-i\omega)\Gamma(1+i\omega)}, \\ |M_-^I|^2 &= \frac{1}{\sin(+i\omega\pi) \sin(-i\omega\pi)} \frac{e^{+\omega\pi}}{\Gamma(1-i\omega)\Gamma(1+i\omega)}. \end{aligned} \quad (54)$$

This means that for all solutions of that type the reflection coefficient always equals to 1:

$$R = 1. \quad (55)$$

Solutions of the second type, rising to infinity as  $z \rightarrow +\infty$ , are characterized by

$$M_+^{II} e^{+i\omega z} + M_-^{II} e^{-i\omega z}, \quad R = \frac{|M_-^{II}|^2}{|M_+^{II}|^2} = e^{4\omega\pi} > 1. \quad (56)$$

Finally, let us specify asymptotic behavior of solutions in terms of Neyman functions. They functions are defined by [37]

$$\begin{aligned} N_{i\omega}(x) &= \frac{\cos(i\omega\pi) J_{i\omega}(x) - J_{-i\omega}(x)}{\sin(i\omega\pi)}, \\ N_{-i\omega}(x) &= \frac{J_{i\omega}(x) - \cos(i\omega\pi) J_{-i\omega}(x)}{\sin(i\omega\pi)}. \end{aligned} \quad (57)$$

In the region  $z \rightarrow +\infty$ , ( $x = iX \rightarrow i\infty$ ), with the use of the known relation [37]

$$N_{i\omega}(x) \sim \sqrt{\frac{2}{i\pi X}} \sin\left(iX - (i\omega + \frac{1}{2})\frac{\pi}{2}\right),$$

we get

$$\begin{aligned} G_1^I(x) &= N_{+i\omega}(x) \sim ie^{+i\pi/4} \sqrt{\frac{1}{2i\pi X}} e^{-\omega\pi/2} e^X, \\ G_1^{II}(x) &= N_{-i\omega}(x) \sim +ie^{+i\pi/4} \sqrt{\frac{1}{2i\pi X}} e^{+\omega\pi/2} e^X. \end{aligned} \quad (58)$$

In the region  $z \rightarrow -\infty$  their behavior is given by

$$\begin{aligned} G^I(z) &= \frac{\cos(i\omega\pi)}{\sin(i\omega\pi)} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z}, \\ G^{II}(z) &= \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{\cos(i\omega\pi)}{\sin(i\omega\pi)} \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z}. \end{aligned} \quad (59)$$

For these solutions we have respectively

$$\begin{aligned} R^I &= \frac{e^{2\omega\pi}}{(e^{2\omega\pi} + e^{-2\omega\pi})/4} = \frac{4}{1 + e^{-4\omega\pi}}, \\ R^{II} &= e^{2\omega\pi} (e^{2\omega\pi} + e^{-2\omega\pi})/4 = \frac{1 + e^{4\omega\pi}}{4}. \end{aligned} \quad (60)$$

## 7 On explicit form of the function $G_2$

The function  $G_1(x)$  satisfies the Bessel equation

$$\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{\omega^2}{x^2} \right) G_1 = 0; \quad (61)$$

the second function  $G_2(x)$  is determined by

$$G_2 = \frac{x}{\omega} \frac{d}{dx} G_1. \quad (62)$$

Solutions of the Bessel equation obey the following recurrent formulas [37]

$$\begin{aligned} x \frac{d}{dx} F_{i\omega} &= i\omega F_{i\omega} - x F_{i\omega+1} , \\ x \frac{d}{dx} F_{-i\omega} &= +i\omega F_{-i\omega}(x) + x F_{-i\omega-1} , \end{aligned} \quad (63)$$

where  $F_{\pm\nu}$  stands for

$$J_{\pm\nu}(x) , \quad H_{\pm\nu}^{(1)}(x) , \quad H_{\pm\nu}^{(2)}(x) , \quad N_{\pm\nu}(x) .$$

Therefore, with the help of (63), one can express  $G_2$  in terms of the known  $G_1$ . For instance,

$$\begin{aligned} G_1^I(x) &= H_{+i\omega}^{(1)}(x) , \quad G_2^I(x) = i H_{+i\omega}^{(1)}(x) - \frac{x}{\omega} H_{i\omega+1}^{(1)}(x) , \\ G_1^{II}(x) &= H_{+i\omega}^{(2)}(x) , \quad G_2^{II}(x) = i H_{+i\omega}^{(2)}(x) - \frac{x}{\omega} H_{i\omega+1}^{(2)}(x) . \end{aligned} \quad (64)$$

Remember that

$$\begin{aligned} F_1^I &= \frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2 , \\ F_2^I &= -\frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2 , \\ f_3^I &= \frac{e^{2z}}{\omega} (-ib F_1^I + ia F_2^I) = \frac{\sqrt{a^2 + b^2}}{i\omega} G_1 . \end{aligned} \quad (65)$$

Let us examine asymptotic behavior of  $G_2$ . Starting with

$$\begin{aligned} H_{i\omega}^{(1)}(x) &= +\frac{i}{\sin(i\omega\pi)} (e^{\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)) , \\ H_{i\omega}^{(2)}(x) &= -\frac{i}{\sin(i\omega\pi)} (e^{-\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)) , \\ H_{i\omega+1}^{(1)}(x) &= +\frac{i}{\sin(i\omega+1)\pi} (e^{-i(i\omega+1)\pi} J_{+i\omega+1}(x) - J_{-(i\omega+1)}(x)) , \\ H_{i\omega+1}^{(2)}(x) &= -\frac{i}{\sin(i\omega+1)\pi} (e^{i(i\omega+1)\pi} J_{+i\omega+1}(x) - J_{-(i\omega+1)}(x)) , \end{aligned} \quad (66)$$

with the help of relations

$$z \rightarrow -\infty, \quad x \rightarrow i0 ,$$

$$J_{+i\omega}(x) \sim \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} , \quad J_{-i\omega}(x) \sim \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} .$$

we get

$$z \rightarrow -\infty, \quad x \rightarrow i0 ,$$

$$\begin{aligned} H_{i\omega}^{(1)} &\sim +\frac{i}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) , \\ H_{i\omega}^{(2)} &\sim -\frac{i}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) , \end{aligned}$$

$$\begin{aligned}
H_{i\omega+1}^{(1)} &\sim \frac{i}{\sin(i\omega+1)\pi} \left( e^{-i(i\omega+1)\pi} \frac{(i\sigma)^{i\omega+1}}{\Gamma(2+i\omega)} e^{i\omega z} e^z - \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z} \right) \\
&\sim -\frac{i}{\sin(i\omega+1)\pi} \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z}, \\
H_{i\omega+1}^{(2)}(x) &\sim \frac{-i}{\sin(i\omega+1)\pi} \left( e^{i(i\omega+1)\pi} \frac{(i\sigma)^{i\omega+1}}{\Gamma(2+i\omega)} e^{i\omega z} e^z - \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z} \right) \\
&\sim \frac{i}{\sin(i\omega+1)\pi} \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z}.
\end{aligned} \tag{67}$$

So we get

$$\begin{aligned}
G_2^I(x) &= -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) \\
&\quad - \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega+1)\pi} \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z}.
\end{aligned} \tag{68}$$

Taking into consideration an identity

$$\begin{aligned}
&\quad - \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega+1)\pi} \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} \\
&= + \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-i\omega}(-i\omega)}{(i\sigma)\Gamma(1-i\omega)} e^{-i\omega z} = -2 \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z}
\end{aligned} \tag{69}$$

one reduces the above relation (68) to the form

$$G_2^I(x) = -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} + \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right). \tag{70}$$

In similar manner one can treat the case

$$\begin{aligned}
G_2^{II} &= \frac{1}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) \\
&\quad + \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega+1)\pi} \frac{(i\sigma)^{-i\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} \\
&= \frac{1}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} + \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right).
\end{aligned} \tag{71}$$

Behavior of these solutions when  $z \rightarrow +\infty$  is governed the relation

$$\begin{aligned}
H_{i\omega}^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[ +i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right], \\
H_{i\omega}^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[ -i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right];
\end{aligned}$$

from whence it follows

$$\begin{aligned}
H_{i\omega}^{(1)}(x) &\sim e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \\
H_{i\omega}^{(2)}(x) &\sim e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X} \\
H_{i\omega+1}^{(1)}(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[ +i \left( iX - \frac{\pi}{2} (i\omega + 1 + \frac{1}{2}) \right) \right] \\
&\sim -i e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \\
H_{i\omega+1}^{(2)}(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[ -i \left( iX - \frac{\pi}{2} (i\omega + 1 + \frac{1}{2}) \right) \right] \\
&\sim i e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}.
\end{aligned} \tag{72}$$

Therefore, we arrive at the formulas

$$\begin{aligned}
G_2^I(x) &= i H_{i\omega}^{(1)}(x) - \frac{x}{\omega} H_{i\omega+1}^{(1)}(x) \\
&\sim i e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X} - \frac{X}{\omega} e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \\
G_2^{II} &= i H_{i\omega}^{(2)}(x) - \frac{x}{\omega} H_{i\omega+1}^{(2)}(x) \\
&\sim i e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X} + \frac{X}{\omega} e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}.
\end{aligned} \tag{73}$$

Evidently, to find asymptotic for  $G_2$ , it is sufficient to make use of the known asymptotic for  $G_1$ . For instance,

$$\begin{aligned}
G_2^I &\sim \frac{1}{\omega} \frac{d}{dz} \frac{i}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) \\
&= -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} + \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right);
\end{aligned} \tag{74}$$

which coincides with (70). It is a superposition of two plane waves with reflection coefficient  $R = 1$ .

## 8 Concluding remarks

In accordance with (39), an equation below

$$\omega^2 = U(z) \quad \omega^2 = (a^2 + b^2)e^{2z_0} \tag{75}$$

determines a critical point  $z_0$  in which behavior of the function  $G_1(x)$  must change dramatically. To such a point  $z_0$  there corresponds

$$x_0 = i\sqrt{a^2 + b^2}e^{z_0} = i\omega. \tag{76}$$

In order to examine behavior of solutions in vicinity of  $x_0$ , it is convenient to introduce a new coordinate

$$x = x_0 + i\omega u, \quad \frac{d}{dx} = \frac{1}{i\omega} \frac{d}{du}; \quad (77)$$

eq. (40) for  $G_1(x)$  assumes the form

$$\left( \frac{d^2}{du^2} + \frac{1}{1+u} \frac{d}{du} - \omega^2 + \frac{\omega^2}{(1+u)^2} \right) G_1 = 0. \quad (78)$$

Close to  $u = 0$ , we have

$$\left( \frac{d^2}{du^2} + \frac{d}{du} \right) G_1 = 0. \quad (79)$$

that is

$$G_1 = e^{Bu}, \quad B^2 + B = 0, \quad B = 0, -1;$$

physically interesting is the choice  $B = -1$ .

To such a critical value  $x_0 = i\omega$ , there correspond

$$\omega = \sqrt{k_1^2 + k_2^2} e^{z_0} \quad \Longrightarrow \quad z_0 = \ln \frac{\omega}{\sqrt{k_1^2 + k_2^2}}; \quad (80)$$

in usual units, this relation reads

$$z_0 = \rho \ln \frac{\omega}{c \sqrt{k_1^2 + k_2^2}}, \quad (81)$$

where  $\rho$  is a curvature radius of the Lobachevsky space.

Let us summarize results.

Lobachevsky geometry simulates a medium with special constitutive relations. The situation is specified in quasi-cartesian coordinates  $(x, y, z)$ . Exact solutions of the Maxwell equations in complex 3-vector  $\mathbf{E} + i\mathbf{B}$  form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field  $U(z) = U_0 e^{2z}$ .

In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient  $R = 1$ ; in electrodynamic context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency  $\omega$ ,  $k_1^2 + k_2^2$ , and the curvature radius  $\rho$  – see (81). See illustrations in Fig. 2,3.



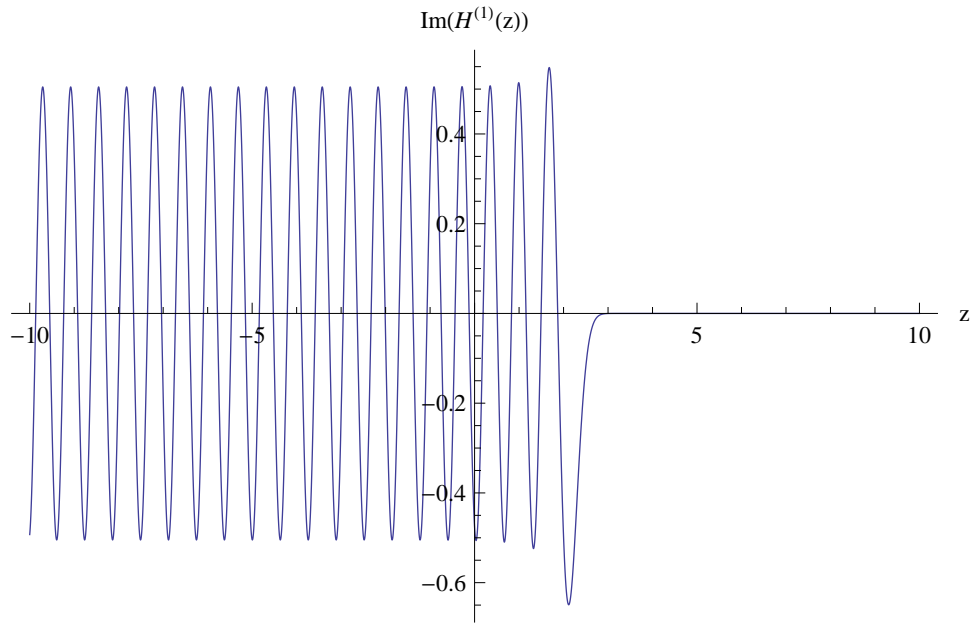


Figure 2:  $\text{Im } H_{+i\omega}^{(1)}, \omega = 10$

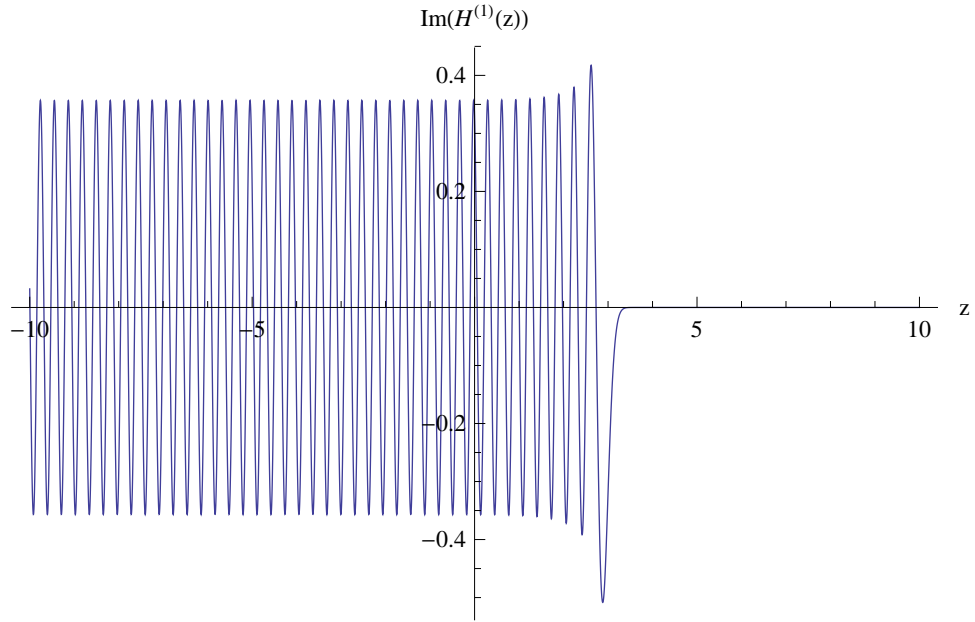


Figure 3:  $\text{Im } H_{+i\omega}^{(1)}, \omega = 20$

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